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Controlled Surgery with Good Local Fundamental Groups

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1. Controlled Surgery Exact Sequence

The aim of this talk is to discuss a possibility to extend the following controlled surgery exact sequence:

Theorem [PQR] (simplified version) *Suppose B is a finite dimensional compact metric ANR, and a dimension $n \geq 4$ is given. Then there exists a number $\epsilon_0 > 0$ which depends on B and n so that for any $\epsilon_0 > \epsilon > 0$ there is $\delta > 0$ with the following property: If $p : X \to B$ is UV^1 and X is a closed topological n-manifold then there is a controlled surgery exact sequence*

$$H_{n+1}(B, \mathbf{L}) \to \mathcal{S}_{\epsilon,\delta}(X, p) \to [X, G/TOP] \to H_n(B, \mathbf{L}) .$$

For $n \geq 5$, it seems that the above should hold true for reasonably good control maps (*e.g.* stratified systems of fibrations) $p : X \to B$, if one replaces the homology groups $H_i(B, \mathbf{L})$ with the controlled L-groups $L_i^c(B, p)$. This may be obvious for experts, but not for me. I will really appreciate it if someone can help me writing down the detailed proof of the controlled surgery exact sequence in this generality.

The reason we have homology when p is UV^1 is that the controlled Whitehead group $Wh^c(B, p)$ vanihes for such p, and this in turn comes from the fact that the ordinary Whitehead groups $Wh(\{1\} \times \mathbf{Z}^j)$ vanish for all $j \geq 0$. Therefore, if all the fundamental groups π of point inverses of p satisfy a similar condition $Wh(\pi \times \mathbf{Z}^j) = 0$ $(\forall j \geq 0)$, then $L_i^c(B,p)$ is isomorphic to a certain generalized homology group $H_i(B, \mathbf{L}(p))$. We say that the local fundamental groups are **good** if this condition is satisfied. The reason we want homology is that we cannot easily compute the controlled L-groups in general.

When the dimension n is equal to 4, we need more assumption. To do anything good in this dimension, the fundamental group has to be also good in the sense of Freedman-Quinn (FQ-good). In the case of controlled surgery the local fundamental groups have to be FQ-good. So we include this in the definition of goodness above.

Typical examples of good local fundamental groups are the free abelian groups $\mathbf{Z}^k$.

As noted above, the key argument of [PQR] seem to work also for the case of good local fundamental groups. But, at this stage, I do not know whether we have the controlled surgery exact sequence for good local fundamental groups or not.

2. 4-dimensional Surgery

Supposing controlled surgery works when the local fundamental groups are good, what can we use it for?

In dimension 4, s-cobordism theorem and surgery theory work if the fundamental group is FQ-good. Groups of subexponential growth are known to be FQ-good. Surprisingly, Krushkal-Lee showed that if X is a 4-dimensional Poincaré complex with a free fundamental group and with an intersection form of a certain special type then a degree one normal map $f : M \to X$ with trivial surgery obstruction in $L_4(\mathbf{Z}\pi_1(X))$ is normally bordant to a homotopy equivalence. In [HR], Hegenbarth and Repovš provide an alternative proof of this and much more using controlled surgery sequence with trivial local fundamental groups. I will briefly discuss their strategy in this section.

Take an element $[f, b] \in [X, G/TOP]$ with trivial surgery obstruction. Pick a UV^1-map $p : X \to B$, and consider the following commutative diagram.

$$\begin{tikzcd}
\mathcal{S}_{\epsilon,\delta}(X,p) \arrow[r] \arrow[d] & {[X, G/TOP]} \arrow[r] \arrow[d, equal] & H_4(B, \mathbf{L}) \arrow[d, "A"] \\
\mathcal{S}(X) \arrow[r] & {[X, G/TOP]} \arrow[r] & L_4(\pi_1(X)).
\end{tikzcd}$$

The first row is known to be exact. We want the second row to be exact. If we assume that the assembly map $A : H_4(B, \mathbf{L}) \to L_4(\pi_1(X))$ is injective, then a diagram chase shows that $[f, b] \in [X, G/TOP]$ comes from an element of $\mathcal{S}(X)$.

Since $H_4(B, \mathbf{L})$ is isomorphic to the controlled L-group $L_4^c(B; p)$, the assumption on A above can be rephrased as follows: Let $c = (C, \psi)$ be a sufficiently controlled 4-dimensional quadratic Poincaré complex on X representing the surgery obstruction for (f, b) and assume that there is an uncontrolled 5-dimensional quadratic Poincaré pair $(g : C \to D, (\delta\psi, \psi))$, then there is a sufficiently cotrolled quadratic Poincaré pair $(g' : C \to D', (\delta\psi', \psi))$.

For various 4-manifolds X, Hegenbarth and Repovš construct UV^1 control maps for which the assembly map A is injective. If the controlled surgery obstruction theory works in the case of good local fundamental groups, then we may be able to find more examples for which the classical surgery obstruction theory works by finding control maps with good local fundamental groups and with injective assembly map.

Question. Is there an analogous trick for proving the triviality of a 5-dimensional s-cobordism when the fundamental group is not FQ-good? More precisely, suppose we have a 5-dimensional s-cobordism, and let C be the relative chain complex giving the torsion of the cobordism. The uncontrolled torsion of C is 0 by assumption. The question is: is there an algebraic criterion for the cobordism to be controlled? If it is controlled, then the cobordism is topologically trivial due to the controlled h-cobordism theorem of Quinn [FQ, p.109], assuming that the local fundamental groups are good. In the case of surgery, the surgery obstruction is always controlled, and the null-cobordism is uncontroled. The condition on A guarantees that the null-cobordism can be controllable as mentioned above. Although there is a similarity, I do not have the answer now.

References

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